Homework 3

check regularly for updates

Due date for HW3 is **Mon, Dec 23, 20:00.**

Rules

[1] File Format: Submit only one PDF file. The filename must follow the format:
Firstname_Lastname_homework3.pdf.
If the file size exceeds Moodle's submission file size limit of 50 MB, the file may be divided into multiple PDFs as needed. In such cases, ensure that the filenames are clearly labeled as follows:
"Firstname_Lastname_homework3_partX.pdf" (X=1,2,3).

[2] Submission Method: Using LaTeX or a handwriting app on an iPad is strongly recommended. If you want to submit handwritten paperwork, it must be scanned by a printer and saved as a PDF. Pictures converted to PDF are not acceptable.

[3]Answer Order: Answer the questions in the same order they are listed. Do not change the order or mix them up.

[4] Highlighting Answers: Underline each final answer clearly.

[5] QuTiP Outputs: When using QuTiP, include only a screenshot of the final result. Do not include the entire code.

[6] Legibility of Handwritten Figures: Handwritten figures must be clear and easy to read. Illegible figures will result in a loss of points.

[7] Deadline: Late submissions will not be accepted under any circumstances.

Each exercise is worth 1 point.
Extra point for each error reported on the forum (minus trivial typos).

A. Vector space for multiple qubits. Tensor product.

So far we have considered a vector space for a single qubit, which is defined by the computational basis ket vectors $|0\rangle$ and $|1\rangle$. Any single-qubit state can be written as a superposition $|\Psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$. We have also considered a vector space for a harmonic oscillator, where the basis contains infinite number of vectors (Fock states, the energy eigenstates), and a general quantum state of an oscillator is written as an infinite sum $|\Psi\rangle = \sum_{n=0}^{\infty} \psi_n|n\rangle$. We have not encountered such an object so far. Describing two or more quantum systems, which are somehow "aware" of each other either due to engineered or unintentional interactions, requires a slightly more sophisticated vector space. Mathematically, constructing a two-qubit vector space requires introducing a new notion of tensor product, usually denoted $\bigotimes$.

Let's start with Dirac notations. We have a qubit A living in its own vector space defined by the computational basis state $|0_A\rangle$ and $|1_A\rangle$ and knowing nothing about qubit B, living in a different vector space, defined by the computational basis states $|0_B\rangle$ and $|1_B\rangle$. The qubit A can be in a superposition state $|\Psi_A\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$. Likewise, the qubit B can be in a superposition state $|\Psi_B\rangle = \beta_0|0\rangle + \beta_1|1\rangle$. We can apply Pauli operators to both qubits, but perhaps we should somehow mark them to know in which vector space they are acting: for example, $\hat{Z}_A|\Psi_A\rangle = \alpha_0|0\rangle - \alpha_1|1\rangle$ while $\hat{Z}_A|\Psi_B\rangle = |\Psi_B\rangle$ (the Pauli operator $\hat{Z}_A$ acts in the vector space of qubit A and does nothing to qubit B). Let us consider a "joint" vector space of qubits A and B by taking the tensor product of their respective computational basis states:

$$|00\rangle \equiv |0_A\rangle \otimes |0_B\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}1\\0\end{pmatrix}\\0\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$|01\rangle \equiv |0_A\rangle \otimes |1_B\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}0\\1\end{pmatrix}\\0\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$|10\rangle \equiv |1_A\rangle \otimes |0_B\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\begin{pmatrix}1\\0\end{pmatrix}\\1\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$|11\rangle \equiv |1_A\rangle \otimes |1_B\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\begin{pmatrix}0\\1\end{pmatrix}\\1\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

The dual vectors and the inner product work as follows:

$$\langle 00|00\rangle = \Big(\langle 0_A|0_A\rangle\Big)\Big(\langle 0_B|0_B\rangle\Big) = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = 1$$

$$\langle 01|00\rangle = \Big(\langle 0_A|0_A\rangle\Big)\Big(\langle 1_B|0_B\rangle\Big) = \begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = 0$$

...

Exercise 1: Verify by examining all the relevant inner products of four-component column and row vectors, that states $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ form an orthonormal set.

So far we have a vector space with an orthonormal basis of four kets, each being a four-component column vector. Next, we need to construct operators acting in this vector space. As usual, it is enough to specify what a given operator does to each of the four basis states. For example, let's consider operator $\hat{X}_A$, which flips the qubit A and does nothing to qubit B and apply it to state $|00\rangle$. In Dirac notations, we get an intuitive answer:

$$\hat{X}_A|00\rangle \equiv \left(\hat{X}|0_A\rangle\right) \otimes |0_B\rangle = |1_A\rangle \otimes |0_B\rangle = |10\rangle$$

In the matrix form, the operator $\hat{X}_A$ is defined as:

$$\hat{X}_A \equiv \hat{X} \otimes \hat{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Let's check that such a matrix works:

$$\left(\hat{X} \otimes \hat{I}\right)|00\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle$$

Exercise 2: Use both Dirac notations and matrix notations to apply the operator $\hat{X} \otimes \hat{I}$ to remaining three basis states $|01\rangle$, $|10\rangle$, $|11\rangle$. Which method do you like better?

Let's repeat the above construction for the operator $\hat{X}_B$, which acts as $\hat{X}$ on qubit B:

$$\hat{X}_B|00\rangle \equiv |0_A\rangle \otimes \left(\hat{X}|0_B\rangle\right) = |0_A\rangle \otimes |1_B\rangle = |01\rangle$$

$$\hat{X}_B \equiv \hat{I} \otimes \hat{X} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\left(\hat{I} \otimes \hat{X}\right)|00\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle$$

Exercise 3: Use both Dirac notations and matrix notations to apply the operator $\hat{I} \otimes \hat{X}$ to remaining three basis states $|01\rangle$, $|10\rangle$, $|11\rangle$.

We can construct more interesting operators, acting on both qubits, such as the "exchange interaction" operator $\hat{X}_A\hat{X}_B$. In Dirac notations, we get:

$$\hat{X}_A\hat{X}_B|00\rangle = \left(\hat{X}|0_A\rangle\right) \otimes \left(\hat{X}|0_B\right) = |1_A\rangle \otimes |1_B\rangle$$

In matrix notation, we have two equivalent ways to figure out the same resulting state:

$$\hat{X}_A\hat{X}_B|00\rangle = (\hat{X} \otimes \hat{I})(\hat{I} \otimes \hat{X})|00\rangle = (\hat{X} \otimes \hat{X})|00\rangle$$

Exercise 4: Show that the regular matrix product of $\hat{X} \otimes \hat{I}$ and $\hat{I} \otimes \hat{X}$ equals to the matrix given by a tensor product $\hat{X}\otimes\hat{X} = \begin{pmatrix} 0\begin{pmatrix}0&1\\1&0\end{pmatrix} & 1\begin{pmatrix}0&1\\1&0\end{pmatrix} \\ 1\begin{pmatrix}0&1\\1&0\end{pmatrix} & 0\begin{pmatrix}0&1\\1&0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$

Exercise 5: Apply operator $\hat{X}_A\hat{X}_B$ to all four computational basis states. Show that it exchanges the states $|01\rangle$ and $10\rangle$, which means that it takes a quantum of energy from qubit A and gives it to qubit B (and vice versa).

Another common operator is the parity operator $\hat{Z}_A\hat{Z}_B$. By analogy with the $\hat{X}_A\hat{X}_B$ operator, we get:
$\hat{Z}_A\hat{Z}_B|00\rangle = +|00\rangle$
$\hat{Z}_A\hat{Z}_B|11\rangle = +|11\rangle$
$\hat{Z}_A\hat{Z}_B|01\rangle = -|01\rangle$
$\hat{Z}_A\hat{Z}_B|10\rangle = -|10\rangle$

Observe that computational states are eigenstates of $\hat{Z}_A\hat{Z}_B$, with the eigenvalue $+1$ for states with "equal" qubits and -1 for states with "opposite" qubits. By contrast, the computational states are not eigenstates of $\hat{X}_A\hat{X}_B$. However, we can find the eigenstates by recalling the single-qubit eigenstates of $\hat{X}$: $\hat{X}|\pm\rangle = \hat{X}\left(\frac{1}{\sqrt{2}}|0\rangle \pm \frac{1}{\sqrt{2}}|1\rangle\right) = \pm|\pm\rangle$, hence
$\hat{X}_A\hat{X}_B\Big(|++\rangle \equiv |+_A\rangle\otimes|+_B\rangle\Big) = \Big(\hat{X}_A|+_A\rangle\Big)\otimes\Big(\hat{X}_B|+_B\rangle\Big) = |+_A\rangle\otimes|+_B\rangle \equiv |++\rangle$
$\hat{X}_A\hat{X}_B|--\rangle = |--\rangle$
$\hat{X}_A\hat{X}_B|+-\rangle = -|+-\rangle$
$\hat{X}_A\hat{X}_B|-+\rangle = -|-+\rangle$
Thus, the operator $\hat{X}_A\hat{X}_B$ is also a parity operator but in a rotated basis.

Exercise 6: Find the column vectors corresponding to $|++\rangle$, $|+-\rangle$, $|-+\rangle$, and $|--\rangle$.
For example, $|++\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) = \frac{1}{\sqrt{2}}(|00\rangle+|01\rangle+|10\rangle+|11\rangle) = \begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}$

Exercise 7: Show by an explicit calculation that the four column vectors above are indeed the eigenvectors of the matrix $\hat{X}\otimes\hat{X}$.

We can consider operators of the form $\hat{Z}_A+\hat{Z}_B$. This is a hermitian operator, it represents a physical observable, the sum of the Z-projections of two spins. For example:

$\Big(\hat{Z}_A+\hat{Z}_B\Big)|10\rangle = \hat{Z}_A|10\rangle + \hat{Z}_B|10\rangle = +1|10\rangle + (-1)|10\rangle = 0$

Exercise 8: Find the matrix for the operator $\hat{Z}_A+\hat{Z}_B$. Check that the computational states are the eigenstates. Find the corresponding eigenvalues.

Now that we know how to apply two-qubit operators to the four basis states, we can do so to more general two-qubit states. For example, let us consider qubit A in a state

$|\Psi_A\rangle = \alpha_0|0_A\rangle + \alpha_1|1_A\rangle$ and likewise for qubit B, $|\Psi_B\rangle = \beta_0|0_B\rangle + \beta_1|1_B\rangle$. We of course keep in mind the usual normalization condition $|\alpha_0|^2 + |\alpha_1|^2 = |\beta_0|^2 + |\beta_1^2 = 1$. Lets work out the tensor product of such single-qubit superposition states.
In Dirac notation, we get:
$|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle =$
$= \Big(\alpha_0|0_A\rangle + \alpha_1|1_A\Big) \otimes \Big(\beta_0|0_B\rangle + \beta_1|1_B\rangle\Big) = \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$
In matrix notation, we get:

$$|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle = \begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \otimes \begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix} = \begin{pmatrix}\alpha_0 \begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}\\ \alpha_1 \begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}\alpha_0\beta_0\\ \alpha_0\beta_1\\ \alpha_1\beta_0\\ \alpha_1\beta_1\end{pmatrix}$$

Exercise 9: Verify that for any two single-qubit states satisfying $\langle\Psi_A|\Psi_A\rangle = \langle\Psi_B|\Psi_B\rangle = 1$, a two-qubit state $|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$ would also satisfy $\langle\Psi|\Psi\rangle = 1$.

Exercise 10: Show that $\Big(\hat{Z}_A\hat{X}_B\Big)\Big(|\Psi_A\rangle \otimes |\Psi_B\rangle\Big) = \Big(\hat{Z}|\Psi_A\rangle\Big) \otimes (\hat{X}|\Psi_B\rangle\Big)$. Use matrix notations.

Product states vs. Entangled states

A two-qubit states of the form $|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$ is called a **product state**. How many real numbers do we need to define a product state for a two-qubit system? In principle, α_0 and α_1 are complex numbers, so we need 4 real numbers to define those. However, we have a normalizaiton constraint $|\alpha_0|^2 + |\alpha_1|^2 = 1$ and the irrelevance of the global phase factor, which means we can choose α_0 to be a real number. Therefore, we only need two real numbers to define a single-qubit state $|\Psi_A\rangle$. Same for $|\Psi_B\rangle$. So the product state $|\Psi_A\rangle \otimes |\Psi_B\rangle$ is defined by 4 real numbers.

Lets take another viewpoint. A general two-qubit state must be written as a superposition of the two-qubit basis states:

$$|\Psi\rangle = \psi_{00}|00\rangle + \psi_{01}|01\rangle + \psi_{10}|10\rangle + \psi_{11}|11\rangle \tag{1}$$

Such a state involves 4 complex numbers, that is 8 real numbers. The normalization condition $\langle\Psi|\Psi\rangle = 1$ introduces one constraint $|\psi_{00}|^2 + |\psi_{01}|^2 + |\psi_{10}|^2 + |\psi_{11}|^2 = 1$. The irrelevance of the global phase factor introduces another constraint, that we can set, say ψ_{00} to be a real number. Only relative phase between the amplitudes matter. With these two constraints we are reduced to 6 real numbers. Which is more than 4 in our product-state counting!

Exercise 11: Consider an N-qubit product state, defined as a tensor product of N single-qubit states, $|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle \otimes |\Psi_C\rangle \otimes ...$ What is the length of the resulting column vector? How many real numbers do we need to describe such an N-qubit state? Hint: it's still 2 real numbers per qubit, right?

Exercise 12: Consider a general N-qubit state of the form

$$|\Psi\rangle = \psi_{00...0}|00...0\rangle + \psi_{00...1}|00...1\rangle + ... \tag{2}$$

The length of the N-qubit columnt vector is the same as in the product state. But how many real numbers do we need to describe this state?
Hint: it's still only two constraints, normalization and the global phase factor.

Exercise 13: The number of atoms in the known Universe is estimates as 10^{80} (talking about astronomically large numbers). Consider a register of $N = 256$ qubits. How many real numbers would we need to store a product state? A general general quantum state? If we use 1 atom to store one real number (which is way to little matter for our current technology), do we have enough in the Universe?

The resolution of the apparent paradoxical discrepancy in parameter counting is that a general quantum state differs from a product state. For example, a state $|\Psi_P\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$ is a product state because we can write:

$$|\Psi_P\rangle = \tfrac{1}{2}|0\rangle \otimes \Big(|0\rangle + |1\rangle\Big) + \tfrac{1}{2}|1\rangle \otimes \Big(|0\rangle + |1\rangle\Big) = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle \otimes |+\rangle.$$

By contrast, seemingly simpler states cannot be written as product states:

$$|\Psi_{E1}\rangle = \tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle$$
$$|\Psi_{E2}\rangle = \tfrac{1}{\sqrt{2}}|01\rangle + \tfrac{1}{\sqrt{2}}|10\rangle$$

Exercise 14: Prove the above statement.
Hint: Consider a tensor product of two arbitrary single-qubit states and try finding the required values of α_0, α_1, β_0, β_1.

The main outcome of our joint vector space construction is the existence of quantum states which cannot be product states. Such states are called entangled. Quantum information processing is based on highly counterintuitive properties of entangled states.

B: Quantum measurement of composite systems

Let's recall that defining a quantum measurement outcome for a single qubit (or a single oscillator, or any other single quantum system) in a state $|\Psi\rangle$ starts from choosing Hermitian operator $\hat{M}$ representing the observable being measured; (i) the reading of the measurement apparatus is one of the eigenvalues m of $\hat{M}$ (always a real number); (ii) the resulting state "collapses" onto the eigenstate of $\hat{M}$ corresponding to the eigenvalue $|m\rangle$ indicated by the apparatus ($\hat{M}|m\rangle = m|m\rangle$); (iii) The probability of such measurement outcome is given by $\langle m|\Psi\rangle|^2$. Because the eigenstates of any Hermitian operator form a basis in the vector space in which the operator acts, such a measurement rule spans all possible outcomes, that is $\sum_{\text{all } m} |\langle m|\Psi\rangle|^2 = 1$.

The same rule applies to composite systems. Let's consider an example of a two-qubit system consisting of qubit A and qubit B. Let's also consider a an example of a measurement operator $\hat{M} = a\hat{Z}_A + b\hat{Z}_B$, where $a > b$. Indeed, the operator $\hat{M}$ is Hermitian, it represents a weighted sum of Z-projections of the two spins. One may imagine a measurement operator like that naturally arises with an apparatus that is sensitive to the Z-projection of a single spin via the magnetic field the spin creates, so if we add a second spin at a slightly higher distance, the apparatus would feel a smaller field from the second spin and hence the total signal should be represented by a weighted sum of the Z-projections. The eigenstates of $\hat{M}$ are conveniently the two-qubit computational basis states, and each corresponds to a unique eigenvalue:

$$\hat{M}|00\rangle = +(a+b)|00\rangle$$
$$\hat{M}|01\rangle = +(a-b)|01\rangle$$
$$\hat{M}|10\rangle = -(a-b)|10\rangle$$
$$\hat{M}|11\rangle = -(a+b)|11\rangle$$

For a general two-qubit state given by Eq. 1, we should obtain the following measurement results. Measuring $\hat{M}$ would provide a reading $(a+b)$ accompanied by the collapse of $|\Psi\rangle$ onto state $|00\rangle$ with probability $|\langle 00|\Psi\rangle|^2 = |\psi_{00}|^2$; likewise, reading $(a-b)$ would be accompanied by the collapse of $|\Psi\rangle$ onto state $|01\rangle$, and the probability of such an outcome would be $|\langle 01|\Psi\rangle|^2 = |\psi_{01}|^2$, etc.

What if $a = b$, though? That's an interesting measurement situation, the magnetic signal from each spin looks the same for the measurement apparatus. States $|00\rangle$ and $|11\rangle$ still correspond to distinct eigenvalues, $2a$ and $-2a$, respectively. If the measurement apparatus reads $M = 2a$, the initial state collapses to $|\Psi_{M=2a}\rangle = |00\rangle$ and the probability of such an outcome is still $|\psi_{00}|^2$. Likewise, if the measurement apparatus reads $M = -2a$, the state collapses to $|\Psi_{M=-2a}\rangle = |11\rangle$, and the probability of such an outcome is $|\psi_{11}|^2$, as usual. What if the apparatus reads $M = 0$, though? Both states $|01\rangle$ and $|10\rangle$ correspond to the same eigenvalue 0. Moreover, the superposition state $\psi_{01}|01\rangle + \psi_{10}|10\rangle$ is also an eigenstate of $\hat{M}$ with eigenvalue 0. So what should be the state to which $|\Psi\rangle$ collapses following the measurement with an reading $M = 0$? The most we learn about the system with such a measurement result is that it's neither in state $|00\rangle$ nor in state $|11\rangle$. But we can't know if it's specifically in state $|01\rangle$ or state $|10\rangle$ or any superposition of the two. So the logic of quantum measurement says that a reading $M = 0$ must be accompanied by a "partial" collapse of the initial state $|\Psi\rangle$ onto such a state $|\Psi_{M=0}\rangle$ which is most consistent with the information extracted about the system:

$$|\Psi_0\rangle = \frac{1}{\sqrt{|\psi_{01}|^2+|\psi_{10}|^2}}\big(\psi_{01}|01\rangle + \psi_{10}|10\rangle\big)$$

The normalizaiton factor $1/\sqrt{|\psi_{01}|^2 + |\psi_{10}|^2}$ is necessary to satisfy $\langle\Psi_{m=0}|\Psi_{m=0}\rangle = 1$. Finally, we let us figure out the probability of reading $M = 0$, which is according to the general rule is $|\langle\Psi_{M=0}|\Psi\rangle|^2 = |\psi_{01}|^2 + |\psi_{10}|^2$. This answer can be interpreted as the sum of probabilities of two independent ways to have $M = 0$: either to be in state $|01\rangle$ or in state $|10\rangle$. Another way to arrive at the above answer is to notice that since the case $M = 0$ is neither $M = 2a$, nor $M = -2a$, the probability of $M = 0$ must be $1 - |\psi_{00}|^2 - |\psi_{11}|^2 = |\psi_{01}|^2 + |\psi_{10}|^2$.

The phenomenon of partial collapse can be utilized to create entangled states. Consider a system of two qubits, one prepared in state $|+_A\rangle$, the other in state $|+_B\rangle$, such that the two-qubit state is $|\Psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$. Let's measure $\hat{M} = \hat{Z}_A + \hat{Z}_B$.

Exercise 15: What is the probability to measure $M = -2$?

Exercise 16: What is the probability to measure neither $M = 2$ nor $M = -2$?

Exercise 17: What is the probability to measure $M = 0$?

Measuring $M = 0$ collapses the initial state $|\Psi\rangle$ onto the state $|\Psi_0\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$, which is an entangled state! Of course, this is only one of the three possible outcomes, so a measurement-based production of an entangled state from a product state is a probabalistic process. However, if we have sufficiently many copies of qubit pairs in the product state $|+_A\rangle \otimes |+_B\rangle$, we can keep trying the measurement of $\hat{Z}_A + \hat{Z}_B$ (starting each time from the state $|+_A\rangle \otimes |+_B\rangle$) until we get the reading $M = 0$. Then we got the entangled state $\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$ with certainty.

Exercise 18: How many times do we have to try the measurement on a product state $|+_A\rangle \otimes |+_B\rangle$, on average, in order to obtain an entangled state?

A more common situation with a multi-qubit system is measuring only qubit A, that is to use a measurement operator such as $\hat{M} = \hat{Z}_A$ or $\hat{M} = \hat{X}_A$, etc. Let's consider a general two-qubit state of the form Eq. 1 and measure hat Z_A, the Z-projection of the first qubit. As usual, we have to identify the orthogonal eigenstates of the measurement operator, in this case $\hat{Z}_A$:

$\hat{Z}_A|00\rangle = +1|00\rangle$
$\hat{Z}_A|01\rangle = +1|01\rangle$
$\hat{Z}_A|10\rangle = -1|10\rangle$
$\hat{Z}_A|11\rangle = -1|01\rangle$

We again face a situation where the measurement operator has degenerate eigenvalues. Therefore, any superposition of $|00\rangle$ and $|01\rangle$ is in fact an eigenstate with an eigenvalue $Z_A = +1$ and any superposition of $|10\rangle$ and $|11\rangle$ is an eigenstate with an eigenvalue $Z_A = -1$. Physically, the above represents a simple fact that the measurement cannot distinguish all four computational basis states: after all, our measurement by definition does not care about qubit B! So reading $Z_A = +1$ must be accompanied by a partial collapse of $|\Psi\rangle$ to state

$$|\Psi_{Z_A=+1}\rangle = |0_A\rangle \otimes \frac{\left(\psi_{00}|0_B\rangle + \psi_{01}|1_B\rangle\right)}{\sqrt{|\psi_{00}|^2 + |\psi_{01}|^2}}$$

Likewise, reading $Z_A = -1$ must be accompanied by a partial collapse of $|\Psi\rangle$ to state

$$|\Psi_{Z_A=-1}\rangle = |1_A\rangle \otimes \frac{\left(\psi_{10}|0_B\rangle + \psi_{11}|1_B\rangle\right)}{\sqrt{|\psi_{10}|^2 + |\psi_{11}|^2}}$$

It is perhaps easier to understand the measurement results by rewriting the initial state

$|\Psi\rangle$ given by Eq. 1 as

$$|\Psi\rangle = \sqrt{|\psi_{00}|^2 + |\psi_{01}|^2}|\Psi_{Z_A=+1}\rangle + \sqrt{|\psi_{10}|^2 + |\psi_{11}|^2}|\Psi_{Z_A=-1}\rangle$$

Indeed, the kets $|\Psi_{Z_A=+1}\rangle$ and $|\Psi_{Z_A=-1}\rangle$ are orthogonal, normalized to a unity, and they are the eigenstates of the measurement operator corresponding to $Z_A = +1$ and $Z_A = -1$, respectively. The probability of measuring $Z_A = +1$ is given by $|\langle\Psi_{Z_A=+1}|\Psi\rangle|^2 = \psi_{00}^2 + |\psi_{01}|^2$ and the probability of measuring $Z_A = -1$ is given by $|\langle\Psi_{Z_A=-1}|\Psi\rangle|^2 = |\psi_{10}|^2 + |\psi_{11}|^2$. In summary, measuring only one qubit leads to a partial collapse of the state onto a superposition state consistent with the extracted information.

Exercise 19: Start with a general state $|\Psi\rangle$ given by Eq. 1 and measure sequentially $\hat{Z}_A$ and then $\hat{Z}_B$. Consider the four possible measurement outcomes for $(Z_A, Z_B) = (1,1); (1,-1); (-1,1); (-1,-1)$. Describe the states after the first and the second measurement in each of the four cases and calculate the probabilities of each outcome following the described above measurement rule.

Exercise 20: Repeat the previous exercise in reverse order, first measure Z_B and then Z_A. Do you expect any change in the probability of the four possible outcomes?

Exercise 21: This time let us consider a measurement operator $\hat{M} = \hat{Z}_A \otimes \hat{Z}_B$. It's called a parity operator and designing an apparatus that would be implementing such a measurement is quite a challenge for a quantum engineer. Suppose we start in a state $|\Psi\rangle = |+_A\rangle \otimes |+_B\rangle$. Describe all possible measurement outcomes (the values of M and the corresponding final states of the system, the probability of each outcome).

Hint: use the same measurement rules logic as for the operator $\hat{M} = \hat{Z}_A + \hat{Z}_B$. You will produce entangled states!

C: Spooky properties of entangled states

Let's first recall some properties of the projections of a single spin averaged over many measurements.

Exercise 22: Consider a general product state $|\Psi_p\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$. We can choose $|\Psi_A\rangle = \alpha_0|0_A\rangle + \alpha_1|1_A\rangle$. Recall the good old Bloch sphere representation of $|\Psi_A\rangle$ and prove that $|\langle\Psi_p|\hat{X}_A|\Psi_p\rangle|^2 + |\langle\Psi_p|\hat{Y}_A|\Psi_p\rangle|^2 + |\langle\Psi_p|\hat{Z}_A|\Psi_p\rangle|^2 = 1$. A spin, on average, is an arrow with length 1.

Hint: use the fact that $\langle\Psi_B|\Psi_B\rangle = 1$

Exercise 23: Prove that the maximal value of $\langle\Psi_p|\hat{X}_A\hat{X}_B|\Psi_p\rangle + \langle\Psi_p|\hat{Y}_A\hat{Y}_B|\Psi_p\rangle + \langle\Psi_p|\hat{Z}_A\hat{Z}_B|\Psi_p\rangle$ is 1 and the minimal value is -1.

Hint: first prove that for a product state $\langle\Psi_p|\hat{X}_A\hat{X}_B|\Psi_p\rangle = \langle\Psi_A|\hat{X}|\Psi_A\rangle\langle\Psi_B|\hat{X}|\Psi_B\rangle$ and then it's just a question of the scalar product of two regular vectors of length 1.

Let us introduce 4 "basis" two-qubit entangled states:
$|B_0\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$
$|B_1\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$
$|B_2\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$
$|B_3\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$
These four states form a basis in the two-qubit vector space, callsed the Bell basis, and any two-qubit state can be shown to be a superposition of the four Bell basis states. For example $|00\rangle = \frac{1}{\sqrt{2}}|B_2\rangle + \frac{1}{\sqrt{2}}|B_3\rangle$.

Exercise 24: Show that the four Bell states are indeed orthogonal to each other and each has a length 1.

Exercise 25: Suppose we used a measurement of $\hat{Z}_A + \hat{Z}_B$ to create an entangle state $|B_1\rangle$. Verify that:
$|B_0\rangle = \hat{Z}_A|B_1\rangle$
$|B_2\rangle = \hat{X}_A\hat{Z}_A|B_1\rangle$
$|B_3\rangle = \hat{X}_A|B_1\rangle$

Exercise 26: Work out a unitary transformation $\hat{U}$ which changes the basis from states $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ to states $|B_0\rangle$, $|B_1\rangle$, $|B_2\rangle$, $|B_3\rangle$. Write down the matrix for $\hat{U}$ in the computational basis.

Hint: basis change works the same way for a two-qubit system as for a single-qubit system. Just the matrices are 4x4 instead of 2x2.

Exercise 27: Prove that $\langle B_0|\hat{X}_A|B_0\rangle = \langle B_0|\hat{Y}_A|B_0\rangle = \langle B_0|\hat{Z}_A|B_0\rangle = 0$. So, on average, all three orthogonal projections of our spin are zero! Can you imagine a random 3D vector with such a property?

Exercise 28: Prove the above but for the operators of qubit B.

Exercise 29: Prove that $\langle B_0|\hat{X}_A\hat{X}_B|B_0\rangle = \langle B_0|\hat{Y}_A\hat{Y}_B|B_0\rangle = \langle B_0|\hat{Z}_A\hat{Z}_B|B_0\rangle = -1$.

The above result means that $\langle B_0|\hat{X}_A\hat{X}_B|B_0\rangle + \langle B_0|\hat{Y}_A\hat{Y}_B|B_0\rangle + \langle B_0|\hat{Z}_A\hat{Z}_B|B_0\rangle = -3$, which is impossible for two classical arrows of length 1! Somehow, on average we know nothing about the projections of each individual spin, but we know that their values are perfectly anticorrelated: if one is +1 then the other one must be -1 and vice versa. The Bloch sphere really does not make sense for entangled states! Instead, the unusually strong correlations can be represented using the so-called Pauli plot, where each bar height equals to the mean value of every possible product of single-qubit operators (see the figure):

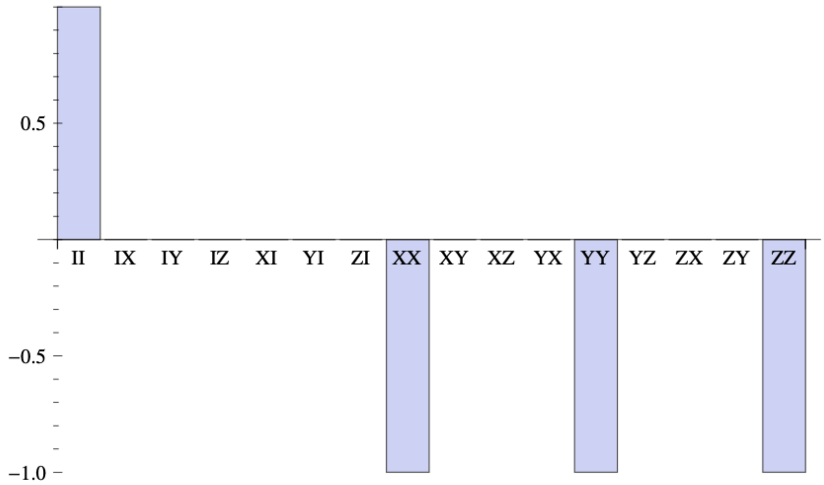

Figure 1: Example Pauli plot for the entangled state $|B_0\rangle$. One can see that the single-qubit operators all have mean values 0, while the product ones all have values of -1. The operator $\hat{I}\hat{I}$ is a trivial case, it is included here for completeness.

Exercise 30: Create the Pauli plot for two states, $|+_A\rangle|-_B\rangle$ and $|B_3\rangle$

Let's go back to the measurement. Suppose we created an entangled state $|B_3\rangle$. Then we gave qubit A to one person, qubit B to another person and flew them to the opposite ends of the planet. Person A can only measure qubit A and apply operations to qubit A; same for person B and their qubit B.

Exercise 31: Show that (i) person A measuring $\hat{Z}_A$ would find $Z_A = +1$ with a probability 1/2 and $Z_A = -1$ also with a probability 1/2. However, every time person A measures $Z_A = +1$, person B afterwards measures $Z_B = +1$ as well.

Exercise 32: Now let's do it such that person A measures $\hat{X}_A$ and right afterwards person B measures $\hat{Z}_B$. Show that now every time person A measures $\hat{X}_A = +1$, person B measures $Z_A = +1$ with a probability 1/2 and $Z_A = -1$ also with a probability 1/2.

Hint: express the state $|B_3\rangle$ using a superposition of tensor products of $|+_A\rangle$, $|-_A\rangle$, $|0_B\rangle$, $1_B\rangle$. Then work out the outcome of measuring $\hat{X}_A$ using the standard quantum measurement rules.

The two examples have been confusing people for a while. It appears that person A can influence the outcome of the measurement done by person B, even though the two can be arbitrary far apart.

D: Deterministic creation of entanglement

We already know that a single spin can feel a magnetic field, and variation of this field can induce the dynamics of the single-qubit quantum state all along the Bloch sphere. Imagine this field is created by a second qubit. This way one qubit can affect the other, and this is the essence of quantum logic. Controlling interaction between qubits is a standard way to create entanglement. Here we consider the simplest qubit-qubit interaction of the form

$$\hat{H} = -\frac{1}{2}\omega_1\hat{Z}_1 - \frac{1}{2}\omega_2\hat{Z}_2 + g\hat{X}_1\hat{X}_2, g \ll \omega_1, \omega_2 \tag{3}$$

Exercise 33: Consider the two-qubit dynamics accordidng to the Hamiltonian Eq. 3. Consider an initial state $|\Psi(t=0)\rangle = |00\rangle, |11\rangle$. Show that for both cases nothing happens. Use $\omega_1 = \omega_2 = 2\pi$ and $g = 2\pi/100$.

Exercise 34: Consider the two-qubit dynamics accordidng to the Hamiltonian Eq. 3. Consider an initial state $|\Psi(t=0)\rangle = |01\rangle, |10\rangle$. Show that these two states oscillate into one another with a period given by $2\pi/g$. Such oscillations are called swaps.

Exercise 35: Consider the initial state $|01\rangle$ and show that at time $t = (1/2) \times 2\pi/g$ the state, up to a normalization factor is $|01\rangle + i|10\rangle$. Likewise, start with $|10\rangle$ and show that that one evolves into $|01\rangle - i|10\rangle$.

Exercise 36: Propose a single-qubit operator which would convert the state $|01\rangle + i|10\rangle$ into one of the Bell states. Just try a few ones that you know...

Since Hamiltonian in Eq. 3 does not do anything to states $|00\rangle$ and $|11\rangle$, we might as well pretend they do not exist in our vector space. What does that mean? That we are back to a 2-component vector space spanned by the states $|\Psi_0\rangle = |01\rangle$ and $|\Psi_1 = 10\rangle$. To solve for the dynamics, we just need to write down the 2x2 matrix for $\hat{H}$ in this new "truncated" basis.

Exercise 37: Express the matrix 2x2 matrix for $\hat{H}$ in Eq. 3 in the basis of states $|01\rangle$ and $|10\rangle$. What combination of Pauli operators is it?

Think of states $|\Psi_0\rangle = 01\rangle$ and $|\Psi_1 = 10\rangle$ as the north and south poles on a Bloch sphere. So we reduced the dynamics of a two-qubit system to that of a single fictitious qubt. This is helpful, because we can more easily find the time evolution of the fictitious qubit and use it to recalculate the resulting two-qubit state. This is quite a general property of multi-qubit system: if the dynamics only involves two basis states out of 4 (or more), we are effectively dealing with a two-level system again.